Chapter 3 Stress

EXAMPLE 3–2

Derive the loading, shear-force, and bending-moment relations for the beam of Figure 3–5*a*.

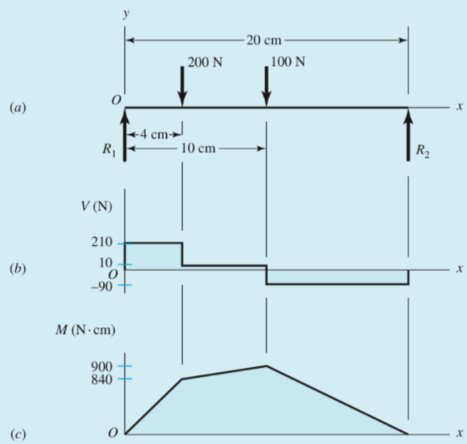


Figure 3–5

(*a*) Loading diagram for a simply-supported beam.
(*b*) Shear-force diagram.
(*c*) Bending-moment diagram.

Solution

Using Table 3–1 and $q(x)$ for the loading function, we find

Answer
$$q = R_1\langle x\rangle^{-1} - 200\langle x-4\rangle^{-1} - 100\langle x-10\rangle^{-1} + R_2\langle x-20\rangle^{-1} \tag{1}$$

Integrating successively gives

Answer
$$V = \int q\,dx = R_1\langle x\rangle^{0} - 200\langle x-4\rangle^{0} - 100\langle x-10\rangle^{0} + R_2\langle x-20\rangle^{0} \tag{2}$$

Answer
$$M = \int V\,dx = R_1\langle x\rangle^{1} - 200\langle x-4\rangle^{1} - 100\langle x-10\rangle^{1} + R_2\langle x-20\rangle^{1} \tag{3}$$

Note that $V = M = 0$ at $x = 0^-$.

The reactions R_1 and R_2 can be found by taking a summation of moments and forces as usual, *or* they can be found by noting that the shear force and bending moment must be zero everywhere except in the region $0 \le x \le 20$ cm. This means that Equation (2) should give $V = 0$ at x slightly larger than 20 cm. Thus

$$R_1 - 200 - 100 + R_2 = 0 \tag{4}$$

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \tag{3–19}$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Shear strain γ is the change in a right angle of a stress element when subjected to pure shear stress, and Hooke's law for shear is given by

$$\tau = G\gamma \tag{3–20}$$

where the constant G is the *shear modulus of elasticity* or *modulus of rigidity*.

It can be shown for a linear, isotropic, homogeneous material, the three elastic constants are related to each other by

$$E = 2G(1+\nu) \tag{3–21}$$

$$\sigma_x = -\frac{My}{I} \tag{3–24}$$

where I is the *second-area moment* about the z axis. That is,

$$I = \int y^2 dA \tag{3–25}$$

The stress distribution given by Equation (3–24) is shown in Figure 3–14. The maximum magnitude of the bending stress will occur where y has the greatest magnitude. Designating $\sigma_{\max}$ as the maximum *magnitude* of the bending stress, and c as the maximum *magnitude* of y

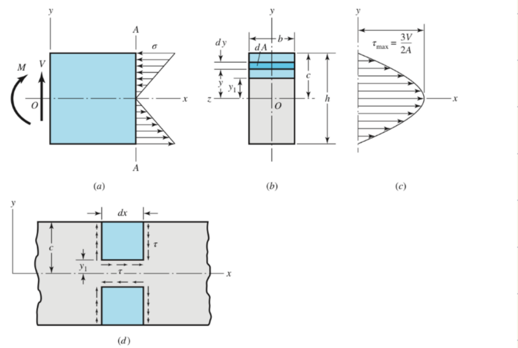
$$\sigma_{\max} = \frac{Mc}{I} \tag{3–26a}$$

Equation (3–24) can still be used to ascertain whether $\sigma_{\max}$ is tensile or compressive.

Equation (3–26a) is often written as

$$\sigma_{\max} = \frac{M}{Z} \tag{3–26b}$$

where $Z = I/c$ is called the *section modulus*.



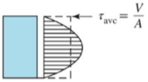
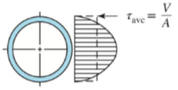
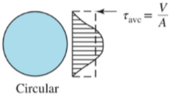
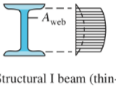
If we now use this value of I for Equation (3–32) and rearrange, we get

$$\tau = \frac{3V}{2A}\left(1 - \frac{y_1^2}{c^2}\right) \tag{3–33}$$

We note that the maximum shear stress exists when $y_1 = 0$, which is at the bending neutral axis. Thus

$$\tau_{\max} = \frac{3V}{2A} \tag{3–34}$$

Table 3–2 **Formulas for Maximum Transverse Shear Stress from VQ/Ib**

Beam Shape	Formula	Beam Shape	Formula
 Rectangular	$\tau_{\max} = \frac{3V}{2A}$	 Hollow, thin-walled round	$\tau_{\max} = \frac{2V}{A}$
 Circular	$\tau_{\max} = \frac{4V}{3A}$	 Structural I beam (thin-walled)	$\tau_{\max} \approx \frac{V}{A_{\text{web}}}$

Two-Plane Bending

Two plane bending

Quite often, in mechanical design, bending occurs in both xy and xz planes. Considering cross sections with one or two planes of symmetry only, the bending stresses are given by

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{3–27}$$

where the first term on the right side of the equation is identical to Equation (3–24), M_y is the bending moment in the xz plane (moment vector in y direction), z is the distance from the neutral y axis, and I_y is the second area moment about the y axis.

For *noncircular* cross sections, Equation (3–27) is the superposition of stresses caused by the two bending moment components. The maximum tensile and compressive bending stresses occur where the summation gives the greatest positive and negative stresses, respectively. For solid *circular* cross sections, all lateral axes are the same and the plane containing the moment corresponding to the vector sum of M_z and M_y contains the maximum bending stresses. For a beam of diameter d the maximum distance from the neutral axis is $d/2$, and from Table A–18, $I = \pi d^4/64$. The maximum bending stress for a solid circular cross section is then

$$\sigma_m = \frac{Mc}{I} = \frac{(M_y^2 + M_z^2)^{1/2}(d/2)}{\pi d^4/64} = \frac{32}{\pi d^3}(M_y^2 + M_z^2)^{1/2} \tag{3–28}$$

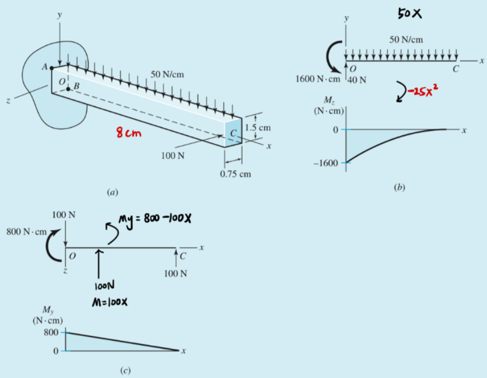
EXAMPLE 3–6

As shown in Figure 3–16*a*, beam *OC* is loaded in the xy plane by a uniform load of 50 N/cm, and in the xz plane by a concentrated force of 100 N at end *C*. The beam is 8 cm long.

$\int_0^L 50x\,dx = 25x^2\big|_0^8$

Figure 3–16

(*a*) Beam loaded in two planes; (*b*) loading and bending-moment diagrams in xy plane; (*c*) loading and bending-moment diagrams in xz plane.



(*a*) For the cross section shown determine the maximum tensile and compressive bending stresses and where they act.

(*b*) If the cross section was a solid circular rod of diameter, $d = 1.25$ cm, determine the magnitude of the maximum bending stress.

Solution

(*a*) The reactions at O and the bending-moment diagrams in the xy and xz planes are shown in Figures 3–16*b* and *c*, respectively. The maximum moments in both planes occur at O where

$$(M_z)_O = -\frac{1}{2}(50)8^2 = -1600 \text{ N}\cdot\text{cm} \qquad (M_y)_O = 100(8) = 800 \text{ N}\cdot\text{cm}$$

The second moments of area in both planes are

$$I_z = \frac{1}{12}(0.75)1.5^3 = 0.2109 \text{ cm}^4 \qquad I_y = \frac{1}{12}(1.5)0.75^3 = 0.05273 \text{ cm}^4$$

The maximum tensile stress occurs at point A, shown in Figure 3–16*a*, where the maximum tensile stress is due to both moments. At A, $y_A = 0.75$ cm and $z_A = 0.375$ cm. Thus, from Equation (3–27)

离质心的距离

Answer
$$(\sigma_x)_A = -\frac{-1600(0.75)}{0.2109} + \frac{800(0.375)}{0.05273} = 11\,380 \text{ Pa} = 11.38 \text{ kPa}$$

The maximum compressive bending stress occurs at point B, where $y_B = -0.75$ cm and $z_B = -0.375$ cm. Thus

Answer
$$(\sigma_x)_B = -\frac{-1600(-0.75)}{0.2109} + \frac{800(-0.375)}{0.05273} = -11\,380 \text{ Pa} = -11.38 \text{ kPa}$$

(*b*) For a solid circular cross section of diameter, $d = 1.25$ cm, the maximum bending stress at end O is given by Equation (3–28) as

Answer
$$\sigma_m = \frac{32}{\pi(1.25)^3}[800^2 + (-1600)^2]^{1/2} = 9329 \text{ Pa} = 9.329 \text{ kPa}$$

$$\theta = \frac{Tl}{GJ} \tag{3–35}$$

where T = torque
l = length
G = modulus of rigidity
J = polar second moment of area

Shear stresses develop throughout the cross section. For a round bar in torsion, these stresses are proportional to the radius ρ and are given by

$$\tau = \frac{T\rho}{J} \tag{3–36}$$

Designating r as the radius to the outer surface, we have

沿圆周切向

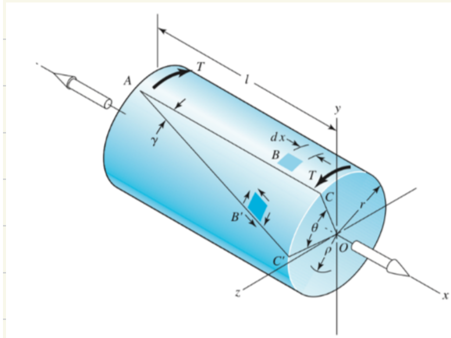
$$\tau_{\max} = \frac{Tr}{J} \tag{3–37}$$

The assumptions used in the analysis are:

- The bar is acted upon by a pure torque, and the sections under consideration are remote from the point of application of the load and from a change in diameter.
- The material obeys Hooke's law.
- Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.

Torque:

$P = T\omega$



The last assumption depends upon the axisymmetry of the member, so it does not hold true for noncircular cross sections. Consequently, Equations (3–35) through (3–37) apply *only* to circular sections. For a solid round section,

$$J = \frac{\pi d^4}{32} \tag{3–38}$$

where d is the diameter of the bar. For a hollow round section,

$$J = \frac{\pi}{32}(d_o^4 - d_i^4) \tag{3–39}$$

where the subscripts o and i refer to the outside and inside diameters, respectively.

When SI units are used, the equation is

$$H = T\omega$$

where H = power, W
T = torque, N · m
ω = angular velocity, rad/s

There are some applications in machinery for noncircular cross-section members and shafts where a regular polygonal cross section is useful in transmitting torque to a gear or pulley that can have an axial change in position. Because no key or keyway is needed, the possibility of a lost key is avoided. The development of equations for stress and deflection for torsional loading of noncircular cross sections can be obtained from the mathematical theory of elasticity. In general, the shear stress does not vary linearly with the distance from the axis, and depends on the specific cross section. In fact, for a rectangular section bar the shear stress is zero at the corners where the distance from the axis is the largest. The maximum shearing stress in a rectangular $b \times c$ section bar occurs in the middle of the *longest* side b and is of the magnitude

$$\tau_{\max} = \frac{T}{\alpha bc^2} \approx \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) \tag{3–40}$$

where b is the width (longer side) and c is the thickness (shorter side). They can *not* be interchanged. The parameter α is a factor that is a function of the ratio b/c as shown in the following table.[5] The angle of twist is given by

$$\theta = \frac{Tl}{\beta bc^3 G} \tag{3–41}$$

where β is a function of b/c, as shown in the table.

b/c	1.00	1.50	1.75	2.00	2.50	3.00	4.00	6.00	8.00	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

Thin-walled tubes

Closed Thin-Walled Tubes ($t \ll r$)[7]

In closed thin-walled tubes, it can be shown that the product of shear stress times thickness of the wall τt is constant, meaning that the shear stress τ is inversely proportional to the wall thickness t. The total torque T on a tube such as depicted in Figure 3–25 is given by

$$T = \int \tau t r \, ds = (\tau t)\int r \, ds = \tau t(2A_m) = 2A_m t\tau$$

where A_m is the *area enclosed by the section median line*. Solving for τ gives

$$\tau = \frac{T}{2A_m t} \tag{3–45}$$

For constant wall thickness t, the angular twist (radians) per unit of length of the tube θ_1 is given by

$$\theta_1 = \frac{TL_m}{4GA_m^2 t} \tag{3–46}$$

where L_m is the *length of the section median line*. These equations presume the buckling of the tube is prevented by ribs, stiffeners, bulkheads, and so on, and that the stresses are below the proportional limit.

EXAMPLE 3–11

Compare the shear stress on a circular cylindrical tube with an outside diameter of 1 cm and an inside diameter of 0.9 cm, predicted by Equation (3–37), to that estimated by Equation (3–45).

Solution

From Equation (3–37),

$$\tau_{\max} = \frac{Tr}{J} = \frac{Tr}{(\pi/32)(d_o^4 - d_i^4)} = \frac{T(0.5)}{(\pi/32)(1^4 - 0.9^4)} = 14.809T$$

From Equation (3–45),

$$\tau = \frac{T}{2A_m t} = \frac{T}{2(\pi 0.95^2/4)0.05} = 14.108T$$

Taking Equation (3–37) as correct, the error in the thin-wall estimate is −4.7 percent.

EXAMPLE 3–10

A welded steel tube is 40 cm long, has a 0.125 cm wall thickness, and a 2.5 cm by 3.6 cm rectangular cross section as shown in Figure 3–26. Assume an allowable shear stress of 11500 kPa and a shear modulus of 11.5(10^6) kPa.

(*a*) Estimate the allowable torque T.

(*b*) Estimate the angle of twist due to the torque.

Solution

(*a*) Within the section median line, the area enclosed is

$$A_m = (2.5 - 0.125)(3.6 - 0.125) = 8.253 \text{ cm}^2$$

and the length of the median perimeter is

$$L_m = 2[(2.5 - 0.125) + (3.6 - 0.125)] = 11.70 \text{ cm}$$

Answer

From Equation (3–45) the torque T is

$$T = 2A_m t\tau = 2(8.253)0.125(11\,500) = 23\,730 \text{ kN}\cdot\text{cm}$$

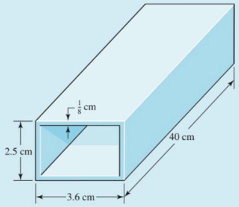
Answer

(*b*) The angle of twist θ from Equation (3–46) is

$$\theta = \theta_1 l = \frac{TL_m}{4GA_m^2 t} l = \frac{23\,730(11.70)}{4(11.5\times 10^6)(8.253^2)(0.125)}(40) = 0.0284 \text{ rad} = 1.62°$$

Figure 3–26

A rectangular steel tube produced by welding.



Example:

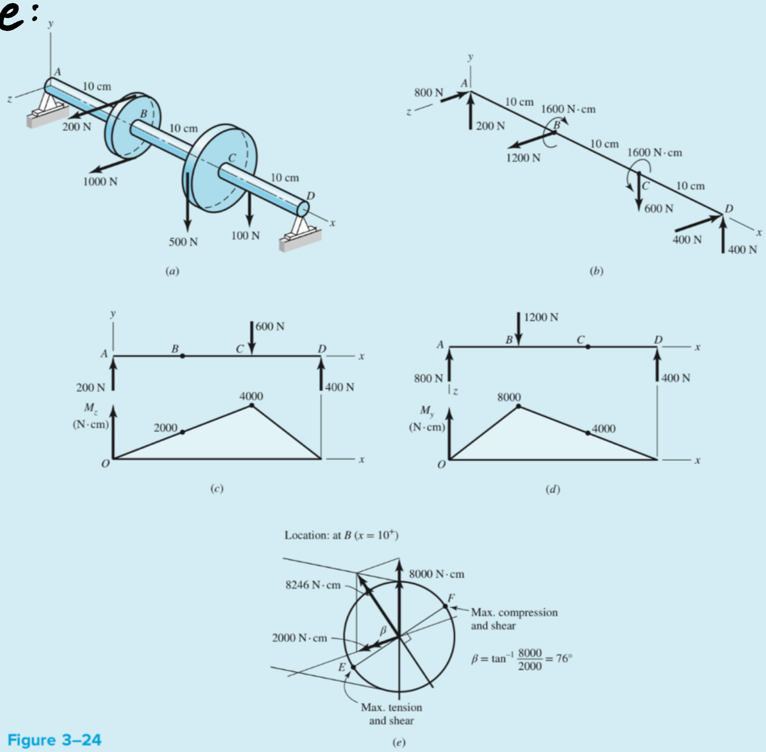


Figure 3–24

The net moment on a section is the vector sum of the components. That is,

$$M = \sqrt{M_y^2 + M_z^2} \tag{1}$$

At point B,

$$M_B = \sqrt{2000^2 + 8000^2} = 8246 \text{ N} \cdot \text{cm}$$

At point C,

$$M_C = \sqrt{4000^2 + 4000^2} = 5657 \text{ N} \cdot \text{cm}$$

Thus the maximum bending moment is 8246 N · cm and the maximum bending stress at pulley B is

$$\sigma = \frac{M\,d/2}{\pi d^4/64} = \frac{32M}{\pi d^3} = \frac{32(8246) \times 10000}{\pi(1.5^3)} = 24890 \times 10^4 \text{ Pa} = 248.9 \text{ MPa}$$

The maximum torsional shear stress occurs between B and C and is

$$\tau = \frac{T\,d/2}{\pi d^4/32} = \frac{16T}{\pi d^3} = \frac{16(1600) \times 10000}{\pi(1.5^3)} = 2414 \times 10^4 \text{ Pa} = 24.14 \text{ MPa}$$

The maximum bending and torsional shear stresses occur just to the right of pulley B at points E and F as shown in Figure 3–24*e*. At point E, the maximum tensile stress will be σ_1 given by

Answer $$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{248.9}{2} + \sqrt{\left(\frac{248.9}{2}\right)^2 + 24.14^2} = 251.2 \text{ MPa}$$

At point F, the maximum compressive stress will be σ_2 given by

Answer $$\sigma_2 = \frac{-\sigma}{2} - \sqrt{\left(\frac{-\sigma}{2}\right)^2 + \tau^2} = \frac{-248.9}{2} - \sqrt{\left(\frac{-248.9}{2}\right)^2 + 24.14^2} = -251.2 \text{ MPa}$$

The extreme shear stress also occurs at E and F and is

Answer $$\tau_1 = \sqrt{\left(\frac{\pm\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{\pm 248.9}{2}\right)^2 + 24.14^2} = 126.8 \text{ MPa}$$

Stress concentration

EXAMPLE 3–13

The 2-mm-thick bar shown in Figure 3–30 is loaded axially with a constant force of 10 kN. The bar material has been heat treated and quenched to raise its strength, but as a consequence it has lost most of its ductility. It is desired to drill a hole through the center of the 40-mm face of the plate to allow a cable to pass through it. A 4-mm hole is sufficient for the cable to fit, but an 8-mm drill is readily available. Will a crack be more likely to initiate at the larger hole, the smaller hole, or at the fillet?

Solution

Since the material is brittle, the effect of stress concentrations near the discontinuities must be considered. Dealing with the hole first, for a 4-mm hole, the nominal stress is

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w-d)t} = \frac{10\,000}{(40-4)2} = 139 \text{ MPa}$$

The theoretical stress concentration factor, from Figure A–15–1, with $d/w = 4/40 = 0.1$, is $K_t = 2.7$. The maximum stress is

Answer

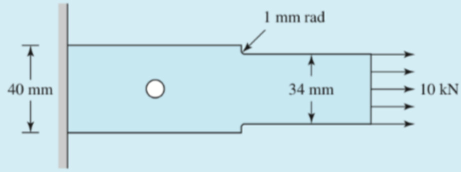
$$\sigma_{\max} = K_t\sigma_0 = 2.7(139) = 375.3 \text{ MPa}$$

Similarly, for an 8-mm hole,

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w-d)t} = \frac{10\,000}{(40-8)2} = 156.25 \text{ MPa}$$

With $d/w = 8/40 = 0.2$, then $K_t = 2.5$, and the maximum stress is

Figure 3–30



Answer

$$\sigma_{\max} = K_t\sigma_0 = 2.5 \times (156.25) = 390.625 \text{ MPa}$$

Though the stress concentration is higher with the 4-mm hole, in this case the increased nominal stress with the 8-mm hole has more effect on the maximum stress.

For the fillet,

$$\sigma_0 = \frac{F}{A} = \frac{10\,000}{(34)2} = 147 \text{ MPa}$$

From Table A–15–5, $D/d = 40/34 = 1.18$, and $r/d = 1/34 = 0.029$. Then $K_t = 2.3$.

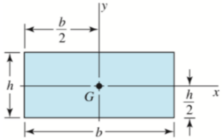
Answer

$$\sigma_{\max} = K_t\sigma_0 = 2.3 \times (147) = 338.2 \text{ MPa}$$

Answer

The crack will most likely occur with the 8-mm hole, next likely would be the 4-mm hole, and least likely at the fillet.

Rectangle



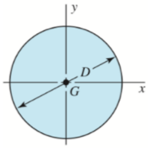
$A = bh \qquad I_x = \frac{bh^3}{12} \qquad I_y = \frac{b^3h}{12} \qquad I_{xy} = 0$

扭转常数 J_t(常用近似，Shigley/工程标准形式)
令 $a=b,\ c=h$ 且 $a \ge c$，则

$$J_t \approx \frac{ac^3}{3}\left[1 - 0.63\left(\frac{c}{a}\right) + 0.052\left(\frac{c}{a}\right)^5\right]$$

(如果 $h>b$，就用 $a=h, c=b$ 套同一式子。)

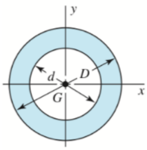
Circle



$A = \frac{\pi D^2}{4} \qquad I_x = I_y = \frac{\pi D^4}{64} \qquad I_{xy} = 0 \qquad J_G = \frac{\pi D^4}{32} \qquad J_t = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$

Hollow circle

$T = 2A_m t\tau$



$J_t = \frac{\pi}{2}(r_o^4 - r_i^4) = \frac{\pi}{32}(d_o^4 - d_i^4)$

$A = \frac{\pi}{4}(D^2 - d^2) \qquad I_x = I_y = \frac{\pi}{64}(D^4 - d^4) \qquad I_{xy} = 0 \qquad J_G = \frac{\pi}{32}(D^4 - d^4)$

4) 空心矩形 (rectangular hollow / box，闭口薄壁更常用

外尺寸 $b \times h$，壁厚 t，内尺寸
$b_i = b - 2t, \qquad h_i = h - 2t$

面积：
$A = bh - b_ih_i = bh - (b-2t)(h-2t)$

二次矩 (外减内，过形心)：
$I_x = \frac{bh^3 - b_ih_i^3}{12}, \qquad I_y = \frac{hb^3 - h_ib_i^3}{12}$

扭转常数 J_t

- **闭口薄壁通用式 (最推荐写法，适用非均匀厚度)**：

$$J_t = \frac{4A_m^2}{\oint \frac{ds}{t}}$$

其中 A_m 为**壁厚中线**围成的面积，$\oint ds/t$ 沿中线周长积分。

- **若壁厚均匀 t(快速版)**：

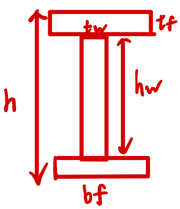
$A_m \approx (b-t)(h-t), \qquad L_m \approx 2[(b-t)+(h-t)]$

$\tau = \frac{T}{2A_m t}$

$J_t \approx \frac{4A_m^2 t}{P_m}$

5) I 形 (I-beam，开口薄壁，常用薄壁近似)

总高 h，翼缘宽 b_f，翼缘厚 t_f，腹板厚 t_w
腹板净高：
$h_w = h - 2t_f$



面积：
$A = 2(b_f t_f) + t_w h_w$

二次矩 (过形心，常用)：

$$I_x = 2\left[\frac{b_f t_f^3}{12} + b_f t_f\left(\frac{h}{2} - \frac{t_f}{2}\right)^2\right] + \frac{t_w h_w^3}{12}$$

$$I_y = 2\left(\frac{t_f b_f^3}{12}\right) + \frac{h_w t_w^3}{12}$$

扭转常数 J_t(开口薄壁近似，最常用)
开口薄壁由若干板段组成：

$$J_t \approx \sum \frac{1}{3} b_i t_i^3$$

对 I 形 (两翼缘 + 一腹板)：

$$J_t \approx \frac{1}{3}\left(2 b_f t_f^3 + h_w t_w^3\right)$$

在传动轴中，主要有：

1 弯曲正应力 (bending normal stress)

$$\sigma_b = \frac{32M}{\pi d^3}$$

2 扭转剪应力 (torsional shear stress)

$$\tau_t = \frac{16T}{\pi d^3}$$

3 横向剪力 V 产生的剪应力

τ_v

但对于圆轴来说：

$$\tau_v = \frac{4V}{3A}$$

这个值通常远小于扭转剪应力 τ_t。

所以在轴设计里：

V 产生的剪应力一般忽略。

6) T 形（T-beam，开口薄壁近似）

翼缘宽 b_f，翼缘厚 t_f；腹板厚 t_w，腹板高 h_w(不含翼缘)，总高 $h = t_f + h_w$

面积：

$A = b_f t_f + t_w h_w$

形心（从翼缘顶面向下量，用于算 I_x）：

$$A_1 = b_f t_f,\ y_1 = \frac{t_f}{2}; \quad A_2 = t_w h_w,\ y_2 = t_f + \frac{h_w}{2}$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

二次矩（过形心）：

$$I_x = \left(\frac{b_f t_f^3}{12} + A_1 (y_1 - \bar{y})^2\right) + \left(\frac{t_w h_w^3}{12} + A_2 (y_2 - \bar{y})^2\right)$$

若腹板在翼缘中线处（关于 y 轴对称）：

$$I_y = \frac{t_f b_f^3}{12} + \frac{h_w t_w^3}{12}$$

扭转常数 J_t(开口薄壁近似)：

$$J_t \approx \frac{1}{3}\left(b_f t_f^3 + h_w t_w^3\right)$$

单位：
F (N)
L (mm)
E / G (MPa = N/mm²)
I (mm⁴)
Jt (mm4)
T (N·mm = 0.001 N·m)
w (rad/s)　$w = \frac{2\pi n}{60}$
n (rpm)
P (kW)
$T(Nm) = \frac{9550 P(kW)}{n(rpm)}$

1 Nm = 1000 N mm

Cross-section selection:

截面	抗弯	抗扭	横向剪力表现	综合评价
实心圆	一般	好	稳定	保守但效率低
空心圆	好	极好	壁薄需注意	高效、各向均匀
实心矩形	中等	差	可接受	结构效率一般
空心矩形	很好	很好	壁薄需注意	弯扭综合最佳
I 形	极好（强轴）	很差	腹板承担	不适合有扭矩
T 形	一般	很差	腹板承担	不适合弯扭组合

好焊接？

Chapter 4: Deflection

The total extension or contraction of a uniform bar in pure tension or compression, respectively, is given by

$$\delta = \frac{Fl}{AE} \qquad \text{(4–3)}$$

l是梁的长度

This equation does not apply to a *long* bar loaded in compression if there is a possibility of buckling (see Sections 4–11 to 4–15). Using Equations (4–2) and (4–3) with $\delta = y$, we see that the spring constant of an axially loaded bar is

$$k = \frac{AE}{l} \qquad \text{(4–4)}$$

The angular deflection of a uniform solid or hollow round bar subjected to a twisting moment T was given in Equation (3–35), and is

(换成角度$\times\frac{180}{\pi}$) 弧度

$$\theta = \frac{Tl}{GJ} \qquad \text{(4–5)}$$

l是扭矩作用的长度

where θ is in radians. If we multiply Equation (4–5) by $180/\pi$ and substitute $J = \pi d^4/32$ for a solid round bar, we obtain

$$\theta = \frac{583.6Tl}{Gd^4} \qquad \text{(4–6)}$$

where θ is in degrees.

Equation (4–5) can be rearranged to give the torsional spring rate as

$$k = \frac{T}{\theta} = \frac{GJ}{l} \qquad \text{(4–7)}$$

Equations (4–5), (4–6), and (4–7) apply *only* to circular cross sections. Torsional loading for bars with noncircular cross sections is discussed in Section 3–12. For the angular twist of rectangular cross sections, closed thin-walled tubes, and open thin-walled sections, refer to Equations (3–41), (3–46), and (3–47), respectively.

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

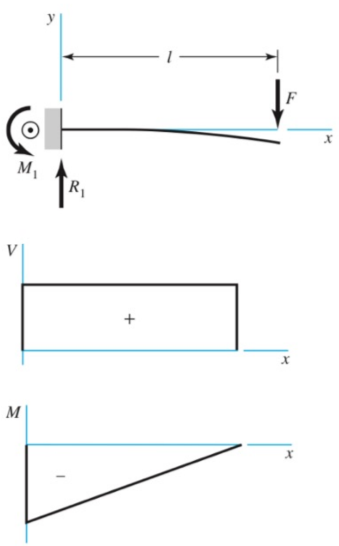
$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\theta = \frac{dy}{dx}$$

$$y = f(x)$$

1 Cantilever—end load



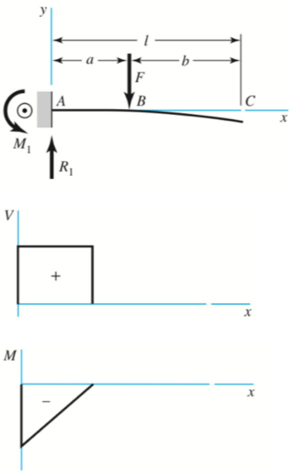
$$R_1 = V = F \qquad M_1 = Fl$$

$$M = F(x - l)$$

$$y = \frac{Fx^2}{6EI}(x - 3l)$$

$$y_{max} = -\frac{Fl^3}{3EI} \qquad (x=l)$$

2 Cantilever—intermediate load



$$R_1 = V = F \qquad M_1 = Fa$$

$$M_{AB} = F(x - a) \qquad M_{BC} = 0$$

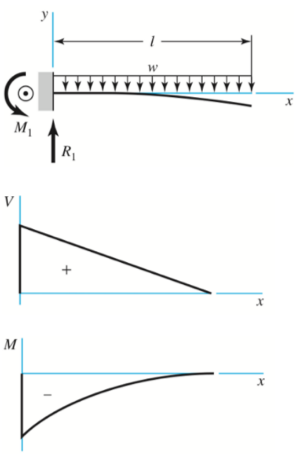
$$y_{AB} = \frac{Fx^2}{6EI}(x - 3a)$$

$$y_{BC} = \frac{Fa^2}{6EI}(a - 3x)$$

$$y_{max} = \frac{Fa^2}{6EI}(a - 3l)$$

$(x=l)$

3 Cantilever—uniform load



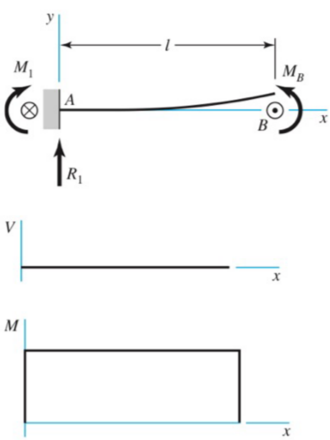
$$R_1 = wl \qquad M_1 = \frac{wl^2}{2}$$

$$V = w(l - x) \qquad M = -\frac{w}{2}(l - x)^2$$

$$y = \frac{wx^2}{24EI}(4lx - x^2 - 6l^2)$$

$$y_{max} = -\frac{wl^4}{8EI}$$ $(x=l)$

4 Cantilever—moment load

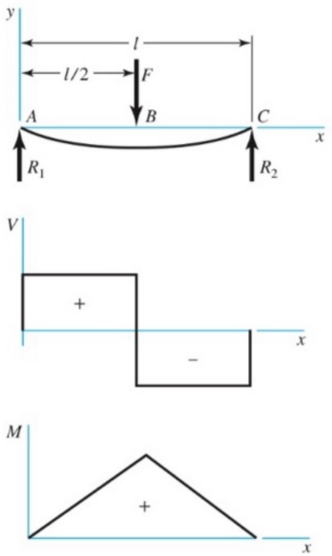


$$R_1 = V = 0 \qquad M_1 = M = M_B$$

$$y = \frac{M_B x^2}{2EI} \qquad y_{max} = \frac{M_B l^2}{2EI}$$

$(x=l)$

5 Simple supports—center load



$$R_1 = R_2 = \frac{F}{2}$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fx}{2} \qquad M_{BC} = \frac{F}{2}(l - x)$$

$$y_{AB} = \frac{Fx}{48EI}(4x^2 - 3l^2)$$

$$y_{\max} = -\frac{Fl^3}{48EI} \quad (x = \frac{l}{2})$$

> **弯矩图的斜率 = 剪力 V(x)**

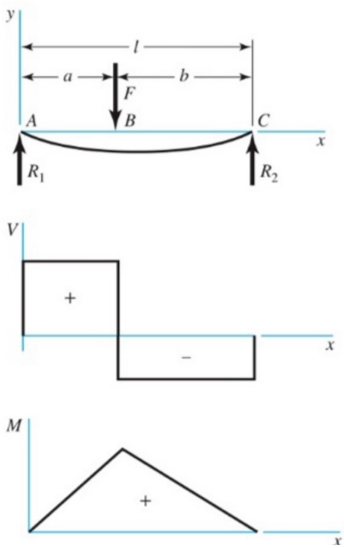
公式是：

$$\frac{dM}{dx} = V(x)$$

所以：

- 剪力为正 → 弯矩斜率为正 → M(x) 上升
- 剪力为负 → 弯矩斜率为负 → M(x) 下降
- 剪力为 0 → 弯矩水平（极值点）

6 Simple supports—intermediate load



$$R_1 = \frac{Fb}{l} \qquad R_2 = \frac{Fa}{l}$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fbx}{l} \qquad M_{BC} = \frac{Fa}{l}(l - x)$$

$$y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l - x)}{6EIl}(x^2 + a^2 - 2lx)$$

$$y_{\max} \left(x = \sqrt{\frac{l^2 - b^2}{3}} \right) \quad x = \frac{l}{2} \ (a = b)$$

假设载荷按位置排序：

$a_1 < a_2 < \dots < a_n$

则：

区间 $0 < x < a_1$

$$M(x) = R_A x$$

区间 $a_1 < x < a_2$

$$M(x) = R_A x - P_1(x - a_1)$$

区间 $a_2 < x < a_3$

$$M(x) = R_A x - P_1(x - a_1) - P_2(x - a_2)$$

(*Continued*)

三、判断斜率的快速口诀

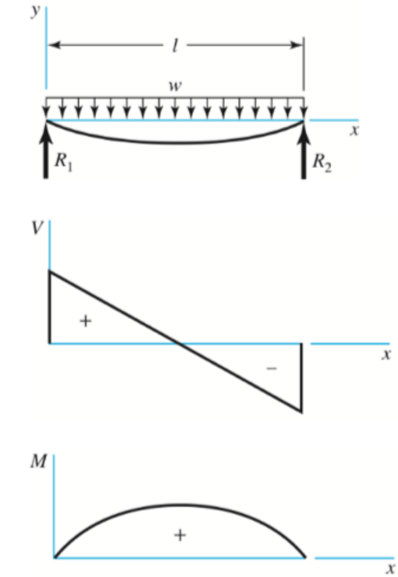
记住一句话：

> 剪力图在上面，弯矩图在下面。
> 剪力正 → 弯矩向上走。
> 剪力负 → 弯矩向下走。

剪力向下为正 M逆时针为正

P向下所以-P

7 Simple supports—uniform load



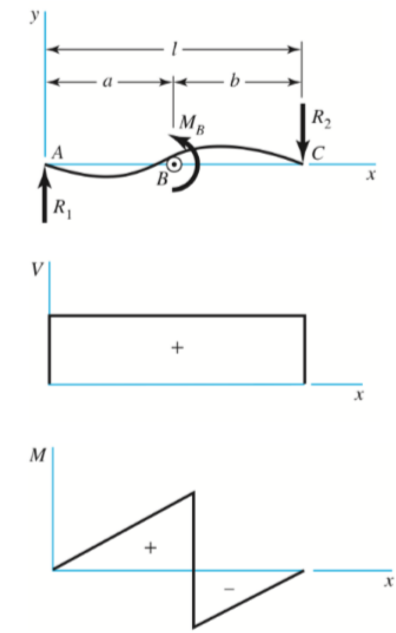
$$R_1 = R_2 = \frac{wl}{2} \qquad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{\max} = -\frac{5wl^4}{384EI} \quad (x = \frac{L}{2})$$

8 Simple supports—moment load



$$R_1 = R_2 = \frac{M_B}{l} \qquad V = \frac{M_B}{l}$$

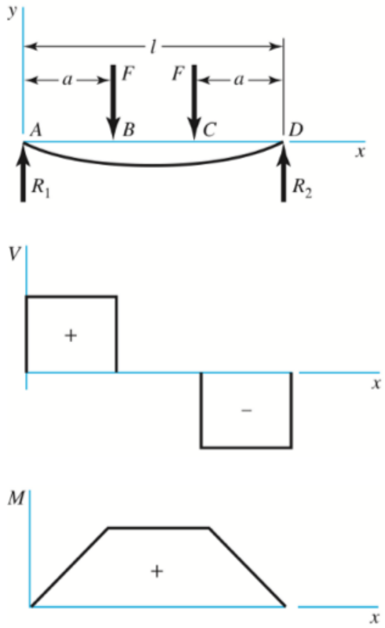
$$M_{AB} = \frac{M_B x}{l} \qquad M_{BC} = \frac{M_B}{l}(x - l)$$

$$y_{AB} = \frac{M_B x}{6EIl}(x^2 + 3a^2 - 6al + 2l^2)$$

$$y_{BC} = \frac{M_B}{6EIl}[x^3 - 3lx^2 + x(2l^2 + 3a^2) - 3a^2 l]$$

$y_{max} \, (x = \frac{L}{2})$

9 Simple supports—twin loads



$$R_1 = R_2 = F \qquad V_{AB} = F \qquad V_{BC} = 0$$

$$V_{CD} = -F$$

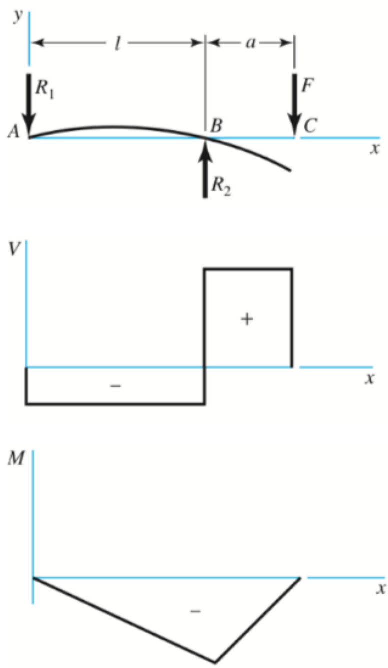
$$M_{AB} = Fx \qquad M_{BC} = Fa \qquad M_{CD} = F(l - x)$$

$$y_{AB} = \frac{Fx}{6EI}(x^2 + 3a^2 - 3la)$$

$$y_{BC} = \frac{Fa}{6EI}(3x^2 + a^2 - 3lx)$$

$$y_{max} = \frac{Fa}{24EI}(4a^2 - 3l^2) \qquad (x = \frac{l}{2})$$

10 Simple supports—overhanging load



$$R_1 = \frac{Fa}{l} \qquad R_2 = \frac{F}{l}(l + a)$$

$$V_{AB} = -\frac{Fa}{l} \qquad V_{BC} = F$$

$$M_{AB} = -\frac{Fax}{l} \qquad M_{BC} = F(x - l - a)$$

$$y_{AB} = \frac{Fax}{6EIl}(l^2 - x^2)$$

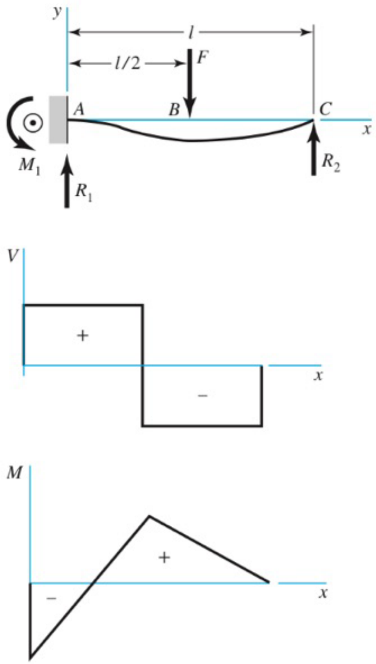
$$y_{BC} = \frac{F(x - l)}{6EI}[(x - l)^2 - a(3x - l)]$$

$$y_C = -\frac{Fa^2}{3EI}(l + a)$$

$y_{max}\ (x = l + a)$

(*Continued*)

11 One fixed and one simple support—center load



$$R_1 = \frac{11F}{16} \qquad R_2 = \frac{5F}{16} \qquad M_1 = \frac{3Fl}{16}$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

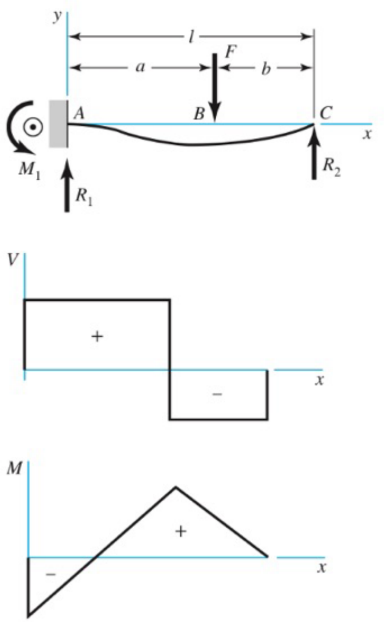
$$M_{AB} = \frac{F}{16}(11x - 3l) \qquad M_{BC} = \frac{5F}{16}(l - x)$$

$$y_{AB} = \frac{Fx^2}{96EI}(11x - 9l)$$

$$y_{BC} = \frac{F(l - x)}{96EI}(5x^2 + 2l^2 - 10lx)$$

$x \approx 0.58l$ (y_{max})

12 One fixed and one simple support—intermediate load



$$R_1 = \frac{Fb}{2l^3}(3l^2 - b^2) \qquad R_2 = \frac{Fa^2}{2l^3}(3l - a)$$

$$M_1 = \frac{Fb}{2l^2}(l^2 - b^2)$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fb}{2l^3}[b^2 l - l^3 + x(3l^2 - b^2)]$$

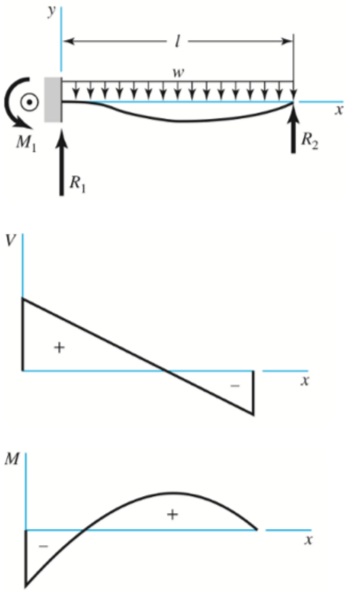
$$M_{BC} = \frac{Fa^2}{2l^3}(3l^2 - 3lx - al + ax)$$

$$y_{AB} = \frac{Fbx^2}{12EIl^3}[3l(b^2 - l^2) + x(3l^2 - b^2)]$$

$$y_{BC} = y_{AB} - \frac{F(x - a)^3}{6EI}$$

$x \approx 0.6l$ (y_{max})

13 One fixed and one simple support—uniform load



$$R_1 = \frac{5wl}{8} \qquad R_2 = \frac{3wl}{8} \qquad M_1 = \frac{wl^2}{8}$$

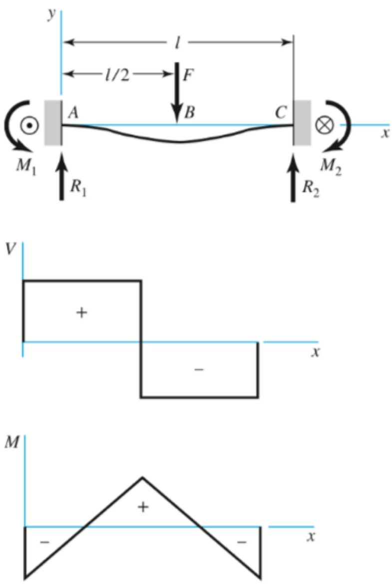
$$V = \frac{5wl}{8} - wx$$

$$M = -\frac{w}{8}(4x^2 - 5lx + l^2)$$

$$y = \frac{wx^2}{48EI}(l - x)(2x - 3l)$$

$x = 0.61 l \ (y_{max})$

14 Fixed supports—center load



$$R_1 = R_2 = \frac{F}{2} \qquad M_1 = M_2 = \frac{Fl}{8}$$

$$V_{AB} = -V_{BC} = \frac{F}{2}$$

$$M_{AB} = \frac{F}{8}(4x - l) \qquad M_{BC} = \frac{F}{8}(3l - 4x)$$

$$y_{AB} = \frac{Fx^2}{48EI}(4x - 3l)$$

$$y_{max} = -\frac{Fl^3}{192EI}$$

$x = \frac{l}{2} \ (y_{max})$

(*Continued*)