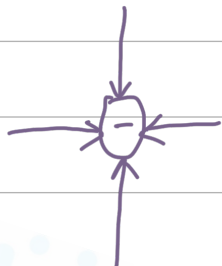
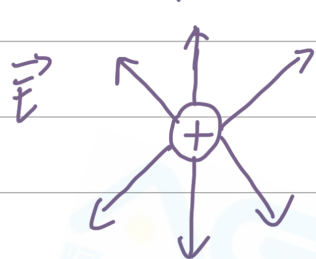


3.7

Coulomb 库仑's law $\vec{F} = \frac{k q_1 q_2}{r^2}$ $k = 9 \times 10^9 \text{ N} \frac{\text{m}^2}{\text{C}^2}$ $k = \frac{1}{4\pi\epsilon_0}$

$$\vec{F}_{1 \rightarrow 2} = \frac{k q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$



强度: density

$$\vec{E} = \frac{\vec{F}}{q} = \frac{kq}{r^2} \hat{r}$$

If the point charge is not at origin

$$\vec{E} = \frac{k q q_0 (\vec{r}^2 - \vec{r}_0^2)}{|\vec{r}^2 - \vec{r}_0^2|^3} = \frac{k q (\vec{r}^2 - \vec{r}_0^2)}{|\vec{r}^2 - \vec{r}_0^2|^3} \quad \vec{E} = \sum_{i=1}^N \frac{k q_i (\vec{r}^2 - \vec{r}_i^2)}{|\vec{r}^2 - \vec{r}_i^2|^3} \text{ Vector sum}$$

3.8 物理 I

total charge

1D Line charge density λ (how much charge per unit length) $\lambda = \frac{dq}{dl}$

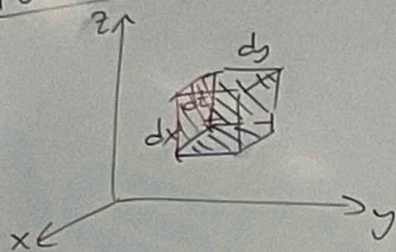
$$dq = \lambda dl \quad \text{total charge } Q_{\text{tot}} = \int dq = \int \lambda dl$$

$$2D \quad \sigma = \frac{dq}{dA} \quad Q_{\text{tot}} = \int dq = \iint \sigma dA = \iint \sigma dx dy \quad \text{double integral}$$

$$3D \quad \rho = \frac{dq}{dv} \quad Q_{\text{tot}} = \int dq = \iiint \rho dv = \iiint \rho dx dy dz \quad \text{triple integral}$$

Coordinate systems

rectangular (x, y, z)

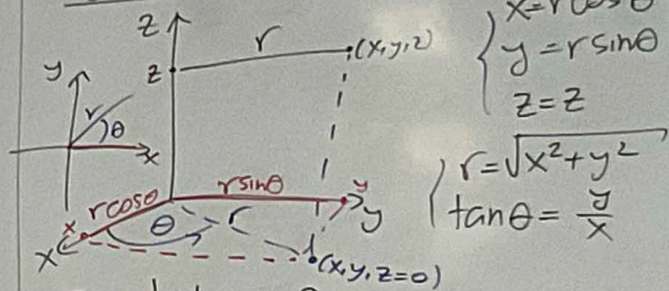


$$dV = dx, dy, dz$$

$$dA = dydz, dxdy, dx dz$$

$$dV = dxdydz$$

Cylindrical - (r, θ, z)



$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

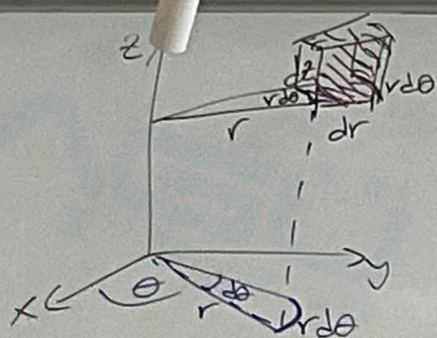
$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \tan \theta = \frac{y}{x} \end{cases}$$

r = distance from z -axis $0 \leq r < \infty$

θ = angle from positive x -axis $0 \leq \theta < 2\pi$

z = height above/below xy plane

$$-\infty < z < \infty$$

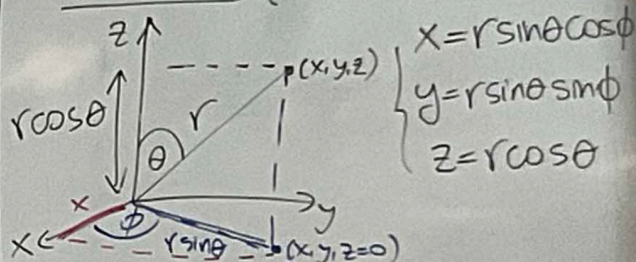


$$dl = dr, dz, r d\theta$$

$$dA = r d\theta dz, r dr, dr dz$$

$$dV = r d\theta dr dz$$

Spherical: (r, ϕ, θ)



r = distance from origin

ϕ = angle from positive x-axis

θ = angle from positive z-axis

$$r = \sqrt{x^2 + y^2 + z^2} \quad 0 \leq r < \infty$$

$$\begin{cases} \tan \phi = \frac{y}{x} & 0 \leq \phi < 2\pi \\ \cos \theta = \frac{z}{r} & 0 \leq \theta \leq \pi \end{cases}$$

均匀分布的球壳 (60)

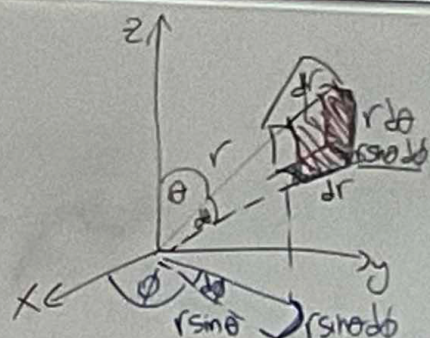
$$q_{\text{shell}} = \int \int b_0 dA$$

$$= b_0 r^2 \sin \theta d\theta d\phi$$

$$= b_0 r^2 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi$$

$$= b_0 r^2 \times (-\cos \pi + \cos 0) \times 2\pi = 4\pi r^2 b_0$$

在外面 ($r > R$) $\rightarrow 4\pi r^2 b_0$



$$d\ell = dr, r d\theta, r \sin \theta d\phi$$

$$dA = r^2 \sin \theta d\theta d\phi, r \sin \theta dr$$

$$dV = r^2 \sin \theta dr d\theta d\phi$$



3.10 物理 I

Electric field of continuous charge distributions

$$\vec{E}(\vec{r}) = \int \frac{k dq (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \begin{cases} \int \frac{k \lambda d\ell (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} & 1D \\ \iint \frac{k b dA (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} & 2D \\ \iiint \frac{k \rho dV (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} & 3D \end{cases}$$

$\vec{r}$: constant

$\vec{r}'$: variable

eg: Electric field of a uniformly charged ring



$$1D: \vec{E}(r) = \int \frac{k \lambda d\ell (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$\vec{r} = (0, 0, z) = z\hat{z}$$

$$\vec{r}' = (R \cos \theta, R \sin \theta, 0)$$

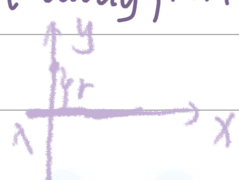
$$d\ell = R d\theta$$

$$E_x = \int_0^{2\pi} \frac{k \lambda R d\theta (-R \cos \theta)}{(R^2 + z^2)^{\frac{3}{2}}} \quad d\theta = 0$$

$$E_y = \int_0^{2\pi} \frac{k\lambda R^2 \sin\theta}{(R^2+z^2)^{\frac{3}{2}}} d\theta = 0$$

$$E_z = \int_0^{2\pi} \frac{k\lambda R d\theta z}{(R^2+z^2)^{\frac{3}{2}}} = \frac{k\lambda R z}{(R^2+z^2)^{\frac{3}{2}}} \int_0^{2\pi} d\theta = \frac{k\lambda R z}{(R^2+z^2)^{\frac{3}{2}}} \cdot 2\pi$$

Above center of a ring

Infinite line of charge $\vec{E}(r) = \frac{2k\lambda}{r} \hat{r}$ 

point charge $\vec{E}(\vec{r}) = \frac{kq(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$ $\vec{r}_0$: 变量 $\vec{r}$: 要求点

eg: insulating disk of radius R using $\int \frac{x^3 dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x^2+2a^2}{\sqrt{x^2+a^2}} + C$
Charged with $\sigma(r) = \sigma_0 r^2$

$$dq = \sigma dA = \sigma_0 r^2 r d\theta dr \quad \vec{r}' = r \cos\theta \hat{x} + r \sin\theta \hat{y} \quad \vec{r} = h \hat{z}$$

$$\vec{r} - \vec{r}' = h \hat{z} - r \cos\theta \hat{x} - r \sin\theta \hat{y} \quad |\vec{r} - \vec{r}'| = \sqrt{h^2 + r^2}$$

Symmetry: $E_x = E_y = 0 \quad \vec{E} = E_z \hat{z}$

$$\vec{E} = \iint \left(\frac{k \sigma_0 r^2 d\theta dr}{(\sqrt{h^2 + r^2})^3} (h \hat{z} - r \cos\theta \hat{x} - r \sin\theta \hat{y}) \right)$$

$$= k \sigma_0 h \hat{z} \int_0^{2\pi} d\theta \int_0^R \frac{r^3 dr}{(h^2 + r^2)^{\frac{3}{2}}}$$

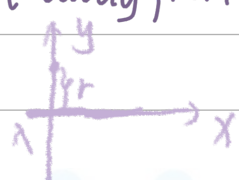
$$= 2\pi k \sigma_0 h \left[\frac{R^2 + 2h^2}{(R^2 + h^2)^{\frac{1}{2}}} - 2h \right] \hat{z}$$

$$E_y = \int_0^{2\pi} \frac{2\lambda R^2 \sin\theta}{(R^2 + z^2)^{\frac{3}{2}}} d\theta = 0$$

$$E_z = \int_0^{2\pi} \frac{kR\lambda d\theta z}{(R^2 + z^2)^{\frac{3}{2}}} = \frac{kR\lambda z}{(R^2 + z^2)^{\frac{3}{2}}} \int_0^{2\pi} d\theta = \frac{k2\pi R\lambda z}{(R^2 + z^2)^{\frac{3}{2}}} \hat{z}$$

Above center of a ring

Infinite line of charge $\vec{E}(r) = \frac{2k\lambda}{r} \hat{r}$ ↑ Point away from the wire



point charge $\vec{E}(\vec{r}) = \frac{kq(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}$ $\vec{r}_0$: 变量 $\vec{r}$: 要求点

eg: insulating disk of radius R using $\int \frac{x^3 dx}{(x^2 + a^2)^{\frac{3}{2}}} = \frac{x^2 + 2a^2}{\sqrt{x^2 + a^2}} + C$
Charged with $\sigma(r) = \alpha r^2$

$$dq = \sigma dA = \alpha r^2 r d\theta dr \quad \vec{r}' = r \cos\theta \hat{x} + r \sin\theta \hat{y} \quad \vec{r} = h \hat{z}$$

$$\vec{r} - \vec{r}' = h \hat{z} - r \cos\theta \hat{x} - r \sin\theta \hat{y} \quad |\vec{r} - \vec{r}'| = \sqrt{h^2 + r^2}$$

Symmetry: $E_x = E_y = 0 \quad \vec{E} = E_z \hat{z}$

$$\vec{E} = \iint \left(\frac{k \alpha r^3 d\theta dr}{(\sqrt{h^2 + r^2})^3} (h \hat{z} - r \cos\theta \hat{x} - r \sin\theta \hat{y}) \right)$$

$$= k \alpha h \hat{z} \int_0^{2\pi} d\theta \int_0^R \frac{r^3 dr}{(h^2 + r^2)^{\frac{3}{2}}}$$

$$= 2\pi \alpha k h \left[\frac{R^2 + h^2}{(R^2 + h^2)^{\frac{1}{2}}} - 2h \right] \hat{z}$$

Since $\vec{E}$ is conservative

$$\Delta U = -W = -\int_{\vec{r}_1}^{\vec{r}_2} q_0 \vec{E} \cdot d\vec{r} = -q_0 \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$$

φ : Electric potential is defined as the potential energy per unit charge $\varphi = \frac{U}{q_0}$

$$\therefore \Delta \varphi = \varphi(\vec{r}_2) - \varphi(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$$

A collection of charge

$$\varphi(\vec{r}) = \sum \frac{kq}{|\vec{r} - \vec{r}'|}$$

A continuous distribution of charge

$$d\varphi(\vec{r}) = \frac{k dq}{|\vec{r} - \vec{r}'|} \quad \varphi(\vec{r}) = \int \frac{k dq}{|\vec{r} - \vec{r}'|} = \begin{cases} 1D \quad dq = \lambda dz & \varphi(\vec{r}) = \int \frac{k \lambda dz}{|\vec{r} - \vec{r}'|} \\ 2D \quad dq = \sigma dA & \varphi(\vec{r}) = \iint \frac{k \sigma dA}{|\vec{r} - \vec{r}'|} \\ 3D \quad dq = \rho dV & \varphi(\vec{r}) = \iiint \frac{k \rho dV}{|\vec{r} - \vec{r}'|} \end{cases}$$

Remark: 1) the equation for the potential assumes that the reference point $r \rightarrow \infty$ $\varphi(r \rightarrow \infty) = 0$

2) We have learned 2 ways

1) By definition $\varphi(\vec{r}) - \varphi(\vec{r}_0) = \int_{r_0}^{\vec{r}} \vec{E} \cdot d\vec{r}$

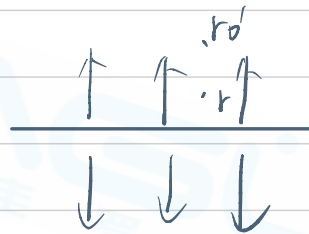
$$2) \varphi(\vec{r}) = \int \frac{k dq}{|\vec{r} - \vec{r}'|} \quad (\vec{r} \rightarrow \infty)$$

In general use Gauss law to find $\vec{E} \rightarrow$ use (1) (symmetry)

Eg. φ of infinite line

Gauss law $E \cdot 2\pi r L = 4\pi k \lambda L \quad E = \frac{2k\lambda}{r} \hat{r}$

$$\varphi(\vec{r}) - \varphi(\vec{r}_0) = - \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{r}$$



$$\begin{aligned} \varphi(\vec{r}) - \varphi(\vec{r}_0) &= - \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{r} = - \int_{\vec{r}_0}^{\vec{r}_0'} \vec{E} \cdot d\vec{r} - \int_{\vec{r}_0'}^{\vec{r}} \vec{E} \cdot d\vec{r} \\ &= - \int_{r_0'}^r \frac{2k\lambda}{r} \hat{r} \cdot dr \hat{r} = - \int_{r_0'}^r \frac{2k\lambda}{r} dr = -2k\lambda \int_{r_0'}^r \frac{1}{r} dr \\ &= -2k\lambda \ln\left(\frac{r}{r_0'}\right) \end{aligned}$$

$E \perp \vec{r} \rightarrow 0$
↑

Eg2 potential of a ring (uniformly charged) (不对称, 法2)

$$\text{Sol: } \vec{r} = (0, 0, z) \quad \vec{r}' = (R \cos \theta, R \sin \theta, 0) \quad dq = \lambda dz = \lambda R d\theta$$

$$|\vec{r} - \vec{r}'| = \sqrt{z^2 + R^2} = \text{const}$$

$$\varphi(0, 0, z) = \int_0^{2\pi} \frac{k \lambda R d\theta}{\sqrt{z^2 + R^2}} = \frac{k \lambda R 2\pi}{\sqrt{z^2 + R^2}} \quad \lambda R 2\pi = Q \quad \varphi(0, 0, z) = \frac{kQ}{\sqrt{z^2 + R^2}}$$

Eg3 Electric potential of a uniformly charged disk

$$\vec{r} = (0, 0, z) \quad \vec{r}' = (r' \cos \theta, r' \sin \theta, 0) \quad dA = r' d\theta dr'$$

$$|\vec{r} - \vec{r}'| = \sqrt{r'^2 + z^2} \quad \varphi = \int_0^R \int_0^{2\pi} \frac{k\sigma r' d\theta dr'}{\sqrt{z^2 + r'^2}} = 2\pi k\sigma \int_0^R \frac{r' dr'}{\sqrt{z^2 + r'^2}} = 2\pi k\sigma \left(\sqrt{z^2 + r'^2} \right) \Big|_{r'=0}^{r'=R}$$

$$= 2\pi k b [\sqrt{2^2 + R^2} - R]$$

Ex4 φ of a uniformly charged spherical shell

① outside $EA = E \times 4\pi r^2 = 4\pi k Q_{in} \quad E = \frac{kQ}{r^2} \hat{r}$

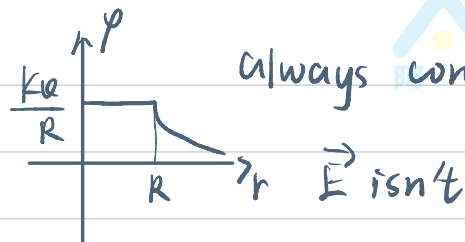
$\varphi(r) - \varphi(r_0) = \int_{r_0}^r \vec{E} \cdot d\vec{r}$ like a point charge $r_0 \rightarrow \infty \quad \varphi(r_0) = 0.$

$$\varphi(r) = \frac{kQ}{r}$$

② $r < R \quad EA = 4\pi k Q_{in} = 0 \quad E = 0 \quad \varphi(r) - \varphi(r_0 \rightarrow \infty) = \int_{\infty}^r \vec{E} \cdot d\vec{r} = - \int_{\infty}^R \vec{E}_{out} \cdot d\vec{r}$

$$\underbrace{- \int_R^r \vec{E}_{in} \cdot d\vec{r}}_0 = \frac{kQ}{R}$$

$$\varphi(r) = \begin{cases} \frac{kQ}{R} & r \leq R \quad \text{constant} \\ \frac{kQ}{r} & r > R \quad \text{like a point charge} \end{cases}$$



Electrostatic energy stored in a system of N point charges

What is the work required to bring all the charges from $r_0 \rightarrow \infty$ to their final position?

The first one comes for free

$q_1 \quad q_2 \quad q_3 \quad \dots \quad \infty$

$$U_{12} = q_2 \cdot \frac{kq_1}{r_{12}} \quad (\text{potential at } q_2 \text{ due to } q_1)$$

$$U_3 = q_3 \cdot \frac{kq_1}{r_{13}} + q_3 \cdot \frac{kq_2}{r_{23}}$$

$$U = \sum_{j>i}^N \sum_{i=1}^N \frac{k q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^N \sum_{\substack{j=1 \\ (i \neq j)}}^N \frac{k q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^N q_i \underbrace{\sum_{\substack{j=1 \\ j \neq i}}^N \frac{k q_j}{r_{ij}}}_{= \varphi_i}$$

potential due to all the charges (except q_i) at the position of q_i

Energy stored in a continuous charge distribution

$$U = \frac{1}{2} \int dq \varphi \quad \varphi \text{ is the potential at the position of } dq$$



$$U = \frac{1}{2} \int dq \varphi = \frac{\epsilon_0}{2} \iiint \vec{E}^2 dv$$

over the
charge body

over all
space

Eg 正方体边长 a 八个角 $(-e)$ 中间 $(2e)$

$$U = \sum_{j>i}^N \sum_{i=1}^N \frac{kq_i q_j}{r_{ij}} = 12 \frac{k(-e)(-e)}{a} + 12 \frac{k(-e)(-e)}{\sqrt{2}a} + 4 \frac{k(-e)(-e)}{\sqrt{3}a}$$

$$+ 8 \frac{(-e)(2e)}{\frac{\sqrt{2}a}{2}}$$

$U > 0$ We need to invest work

$U < 0$ the system wants to stay together

Eg: ... in a spherical thin shell total Q . uniform charge

$$U = \frac{1}{2} \int dq \psi(r=R) = \frac{1}{2} \int dq \frac{kQ}{R} = \frac{1}{2} \frac{kQ}{R} \int dq = \frac{1}{2} \frac{kQ^2}{R}$$

Second approach $U = \frac{\epsilon_0}{2} \left[\int_{r < R} \rho^2 dv + \int_R^{\infty} \left(\frac{kQ}{r^2} \right)^2 4\pi r^2 dr \right] \quad k = \frac{1}{4\pi\epsilon_0}$

$$= \frac{kQ^2}{2} \left[-\frac{1}{r} \right]_R^{\infty} = \frac{kQ^2}{2R}$$

Equipotential surfaces = surfaces connecting points having the same value of electric potential

of between any equipotential surface and its neighbour has the same

value of $\phi \Rightarrow$ more dense equipotential surfaces $\Rightarrow$ stronger $\vec{E}$

$\vec{E}$ is always perpendicular to equipotential surfaces.

Calculating the field from the potential

$$\varphi(\vec{r}) - \varphi(r_0) = - \int_{r_0}^{\vec{r}} \vec{E} \cdot d\vec{r} \Rightarrow E(x) = \frac{-d\varphi}{dx} \Rightarrow \vec{E} = - \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) = -\vec{\nabla} \varphi$$

eg: $\varphi(x, y, z) = x^3 y + z$ $\vec{E}_x = -3yx^2$ $\vec{E}_y = -x^3$ $\vec{E}_z = -1$

$|\vec{\nabla} \varphi|$: magnitude of the maximum slope

$\vec{E} = -\vec{\nabla} \varphi$: $\vec{E}$ points in the direction in which φ decreases more quickly

Electric fields and electric potential around conductors

Conductor: charge can flow inside the conductor.

Electrostatic equilibrium state: charge doesn't move anymore

For an ideal conductor in an electrostatic equilibrium state

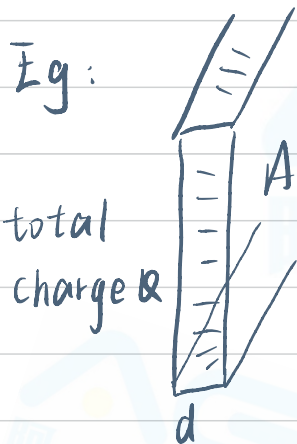
Rule ① $\vec{E} = 0$ inside the material (because if $\vec{E} \neq 0$, the charge will move)

② $\rho = 0$ inside the conductor ($Q_{in} = 0$)

③ $\varphi = \text{constant}$ in the conductor

④ very near the surface on the outer side $\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$

Conductor with a cavity inside $\rightarrow$ Faraday's cage



Find ① charge distribution ② $\vec{E}$ everywhere

Reminder: infinite plane 是常数

